25 Asymptotic Rays and Triangles

Definition (open triangle (or a biangle)). Let A, B, C, D be four points in a neutral geometry, such that no three are collinear, with C and D on the same side of $\overleftrightarrow{AB}$, and $\overleftrightarrow{AD} \| \overleftrightarrow{BC}$. Then the set $\triangle DABC = \overrightarrow{AD} \cup \overline{AB} \cup \overrightarrow{BC}$ is is an open triangle (or a biangle).

Definition (strictly asymptotic). Let $\triangle DABC$ be an open triangle. $\overrightarrow{BC}$ is strictly asymptotic to $\overrightarrow{AD}$ if for every $E \in \text{int}(\measuredangle ABC)$, $\overrightarrow{BE}$ intersects $\overrightarrow{AD}$.

Definition (equivalent, asymptotic). Two rays $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are equivalent (written $\overrightarrow{PQ} \sim \overrightarrow{RS}$) if either $\overrightarrow{PQ} \subseteq \overrightarrow{RS}$ or $\overrightarrow{RS} \subseteq \overrightarrow{PQ}$. The ray $\overrightarrow{BC}$ is asymptotic to the ray $\overrightarrow{AD}$ (written $\overrightarrow{BC} | \overrightarrow{AD}$) if either $\overrightarrow{BC}$ is strictly asymptotic to $\overrightarrow{AD}$ or $\overrightarrow{BC} \sim \overrightarrow{AD}$.

1. Prove that $\sim$ is an equivalence relation on the set of rays in a metric geometry.

Definition (asymptotic (or closed) triangle). The open triangle $\triangle DABC$ is called an asymptotic (or closed) triangle if $\overrightarrow{AD} | \overrightarrow{BC}$.

Definition (asymptotically parallel). Two lines ℓ and ℓ' are asymptotic, or asymptotically parallel (written $\ell | \ell'$), if there are rays $\overrightarrow{AD} \subseteq \ell$ and $\overrightarrow{BC} \subseteq \ell'$ with $\overrightarrow{AD} | \overrightarrow{BC}$.

2. Prove that in a neutral geometry which satisfies EPP, $\ell \| \ell'$ if and only if $\ell | \ell'$.

3. Let $\{\mathcal{S}, \mathcal{L}, d, m\}$ be a neutral geometry such that whenever $\ell_1 \| \ell_2$ then there is a line ℓ' perpendicular to both ℓ_1 and ℓ_2. Prove that EPP is satisfied.

4. Let $\triangle DABC$ be an open triangle. What should be the definition of the interior of $\triangle DABC$? Show that $\text{int}(\triangle DABC)$ is convex.

5. In a neutral geometry, suppose that $\measuredangle DABC$ is an asymptotic triangle. If $\ell \cap \text{int}(\triangle DABC) \neq \emptyset$, prove that $\ell \cap \triangle DABC \neq \emptyset$.

6. In a neutral geometry, if $\overleftrightarrow{AB} \| \overleftrightarrow{CD}$, $\overleftrightarrow{CD} \| \overleftrightarrow{EF}$ and $A - C - E$ prove that $\overleftrightarrow{AB} \| \overleftrightarrow{EF}$.

7. Let $A(0,1)$ and $D(0,2)$. Sketch two different asymptotic triangles $\measuredangle DABC$ in $\mathcal{H}$ for some choices of B and C. How many are there? If $E(1,1)$ find the unique ray $\overrightarrow{EF}$ with $\overrightarrow{EF} | \overrightarrow{AD}$. (See 25.4)

8. In the Poincaré Plane let $A(1,1)$ and $B(1,5)$. (a) Sketch five rays asymptotic to $\overrightarrow{AB}$; (b) Sketch five rays asymptotic to $\overrightarrow{BA}$.

IMPORTANT RESULTS (Asymptotic Rays and Triangles)

(25.1) In a neutral geometry if $\overrightarrow{BC} \sim \overrightarrow{B'C'}$, and $\overrightarrow{BC} | \overrightarrow{AD}$, then $\overrightarrow{B'C'} \sim \overrightarrow{AD}$.

(25.2) In a neutral geometry if $\overrightarrow{AD} \sim \overrightarrow{A'D'}$, and $\overrightarrow{BC} | \overrightarrow{AD}$, then $\overrightarrow{BC} \sim \overrightarrow{A'D'}$.

(25.3) In a neutral geometry if $\overrightarrow{AD} \sim \overrightarrow{A'D'}$, $\overrightarrow{BC} | \overrightarrow{B'C'}$, and $\overrightarrow{BC} \sim \overrightarrow{AD}$ then $\overrightarrow{B'C'} \sim \overrightarrow{A'D'}$.

(25.4) In a neutral geometry, given a ray $\overrightarrow{AD}$ and a point $B \notin \overleftrightarrow{AD}$, there is a unique ray $\overrightarrow{BC}$ with $\overrightarrow{BC} | \overrightarrow{AD}$.

(25.5) In a neutral geometry, if $\overrightarrow{BC} | \overrightarrow{AD}$ then $\overrightarrow{AD} | \overrightarrow{BC}$ also.

(25.6) Let $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$ be distinct lines in a neutral geometry. If $\overrightarrow{AB} | \overrightarrow{CD}$ and $\overrightarrow{CD} | \overrightarrow{EF}$ then there is a line ℓ which intersects all three lines $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$.

(25.7) In a neutral geometry, if $\overrightarrow{AB} | \overrightarrow{CD}$ and $\overrightarrow{CD} | \overrightarrow{EF}$ then $\overrightarrow{AB} | \overrightarrow{EF}$.

(25.8) **(Congruence Theorem for Asymptotic Triangles)** In a neutral geometry, if $\triangle DABC$ and $\triangle SPQR$ are two asymptotic triangles with $\overline{AB} \cong \overline{PQ}$ and $\measuredangle ABC \cong \measuredangle PQR$, then $\measuredangle BAD \cong \measuredangle QPS$.

(25.9) In a neutral geometry which satisfies HPP, if two distinct lines ℓ and ℓ' have a common perpendicular, then the lines are parallel but not asymptotic.